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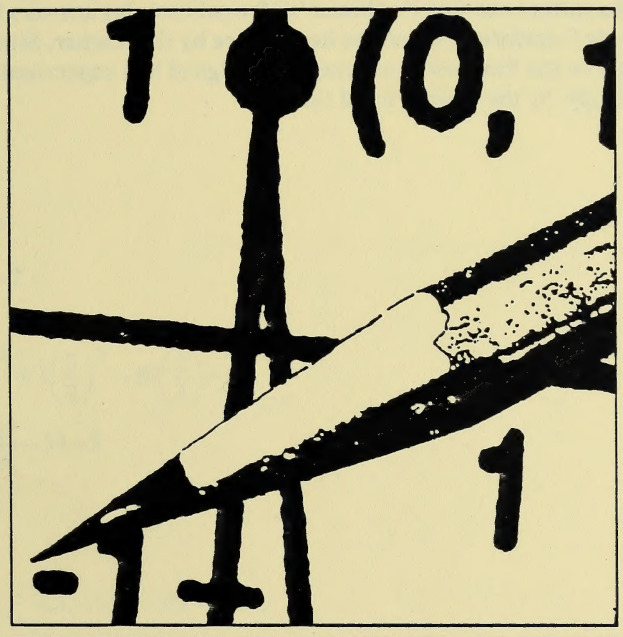
MATHEMATICS 31

LEARNING FACILITATOR'S MANUAL



UNIT 3: PRACTICAL APPLICATION OF DERIVATIVES

Distance Learning

Alberta EDUCATION

Note

This Mathematics Learning Facilitator's Manual contains answers to teacher-assessed assignments and the final test; therefore, it should be kept secure by the teacher. Students should not have access to these assignments or the final tests until they are assigned in a supervised situation. The answers should be stored securely by the teacher at all times.

Mathematics 31
Learning Facilitator's Manual
Unit 3
Practical Application of Derivatives
Alberta Distance Learning Centre
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Topic 1: Maxima and Minima

(7) 1. Find the turning points of the curve $y = 4x^3 + 3x^2 - 36x + 8$.

$$\frac{dy}{dx} = 12x^2 + 6x - 36$$

Let $\frac{dy}{dx} = 0$.

$$12x^2 + 6x - 36 = 0$$
$$6(2x-3)(x+2) = 0$$

$$2x - 3 = 0, \quad x + 2 = 0$$
$$x = \frac{3}{2} \qquad x = -2$$

$$\text{At } x = \frac{3}{2},\ y = 4\left(\frac{3}{2}\right)^3 + 3\left(\frac{3}{2}\right)^2 - 36\left(\frac{3}{2}\right) + 8$$
$$= \frac{27}{2} + \frac{27}{4} - 54 + 8$$
$$= -25\frac{3}{4}$$

$$\text{At } x = -2,\ y = 4(-2)^3 + 3(-2)^2 - 36(-2) + 8$$
$$= -32 + 12 + 72 + 8$$
$$= 60$$

Therefore, the turning points are $\left(\frac{3}{2},\ -25\frac{3}{4}\right)$ and $(-2, 60)$.

⑦ 2. Determine the maximum and/or minimum values of the function $y = 2x(4x^2 - 27x - 30)$.

$y = 8x^3 - 54x^2 - 60x$

$D_x y = 24x^2 - 108x - 60 = 0$

$4x^2 - 18x - 10 = 0$

$2x^2 - 9x - 5 = 0$

$(x-5)(2x+1) = 0$

$x - 5 = 0, \quad 2x + 1 = 0$

$x = 5 \qquad x = -\frac{1}{2}$

At $x = 5$, $y = 8(5)^3 - 54(5)^2 - 60(5)$

$= 1000 - 1350 - 300$

$= -650$

At $x = -\frac{1}{2}$, $y = 8\left(-\frac{1}{2}\right)^3 - 54\left(-\frac{1}{2}\right)^2 - 60\left(-\frac{1}{2}\right)$

$= -1 - -13\frac{1}{2} + 30$

$= 15\frac{1}{2}$

$D_x^2 y = 48x - 108$

At $x = 5$, $D_x^2 y = 48(5) - 108$

$= 132$ (positive)

Therefore, (5, – 650) is a minimum point.
Thus, – 650 is a minimum value.

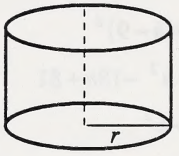
At $x = -\frac{1}{2}$, $D_x^2 y = 48\left(-\frac{1}{2}\right) - 108$

$= -132$ (negative)

Therefore, $\left(-\frac{1}{2}, 15\frac{1}{2}\right)$ is a maximum point.

Thus, $15\frac{1}{2}$ is a maximum value.

⑦ 3. A can of peanuts has a circular base and top, and it is cylindrical in shape. The volume is $500\ \text{cm}^3$. If the can is to require the least amount of material, what must its dimensions be?



$$V = \pi r^2 h$$
$$500 = \pi r^2 h$$
$$h = \frac{500}{\pi r^2}$$

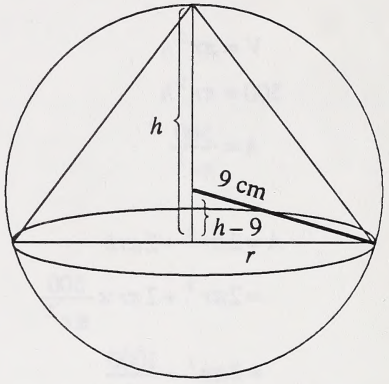
$$A = 2\pi r^2 + 2\pi rh$$
$$= 2\pi r^2 + 2\pi r \times \frac{500}{\pi r^2}$$
$$= 2\pi r^2 + \frac{1000}{r}$$

$$D_r A = 4\pi r - \frac{1000}{r^2} = 0$$
$$4\pi r^3 = 1000$$
$$r^3 = \frac{1000}{4\pi}$$
$$= \frac{250}{\pi}$$
$$r \doteq 4.3$$

$$h \doteq \frac{500}{\pi(4.3)^2}$$
$$\doteq 8.6$$

The radius is approximately 4.3 cm, and the height is approximately 8.6 cm.

(7) 4. Find the maximum volume of the largest right circular cone that can be inscribed in a sphere of radius 9 cm.



$$V = \frac{1}{3}\pi r^2 h$$

$$9^2 = r^2 + (h-9)^2$$

$$= r^2 + h^2 - 18h + 81$$

$$r^2 = 18h - h^2$$

$$\therefore V = \frac{1}{3}\pi\left(18h - h^2\right)h$$

$$= \frac{1}{3}\pi\left(18h^2 - h^3\right)$$

$$D_h V = \frac{1}{3}\pi\left(36h - 3h^2\right) = 0$$

$$h(36 - 3h) = 0$$

$$h = 0 \quad \text{(Discard.)}$$

$$36 - 3h = 0$$

$$h = 12$$

$$r^2 = 18(12) - (12)^2$$

$$= 72$$

$$r = \sqrt{72}$$

$$\doteq 8.49$$

$$\text{Maximum volume} = \frac{1}{3}\pi(72)(12)$$

$$= 288\pi$$

The maximum volume is 288π cm^3.

(7) 5. A toy manufacturer figures that it costs \$4 to make a toy car. There are fixed costs of \$500 per week. The revenue this business generates per week is given by the formula $R = 700x - 0.25x^2$, where x is the number of toy cars produced. For what value of x is the weekly profit a maximum?

Profit = revenue – cost

$$P = (700x - 0.25x^2) - (500 + 4x)$$

$$= 696x - 0.25x^2 - 500$$

$$D_x P = 696 - 0.5x = 0$$

$$x = 1392$$

Weekly profit is at a maximum when 1392 toy cars are produced.

Topic 1

_____ marks

Topic 2: Implicit Differentiation and Motion Problems

1. An object is moving in a straight line from a fixed point. The displacement (s) in metres is given by $s = -2t^2 + 28t + 45$, $t \geq 0$, where t is time in seconds.

(2) a. Find the velocity at any time.

$v = D_t s = -4t + 28$

(2) b. What is the velocity at $t = 3$?

At $t = 3$, $v = -4(3) + 28$

$= 16$

The velocity at $t = 3$ is 16 m/s.

(2) c. Find t when the object reaches its maximum displacement.

$-4t + 28 = 0$

$4t = 28$

$t = 7$

The time is 7 s.

(2) d. Find the maximum displacement reached by this object.

At $t = 7$, $s = -2(7)^2 + 28(7) + 45$

$= -98 + 196 + 45$

$= 143$

The maximum displacement is 143 m.

(2) e. What is the average velocity from $t = 0$ to $t = 8$?

At $t = 0$, $s = 45$

At $t = 8$, $s = -2(8)^2 + 28(8) + 45$

$= -128 + 224 + 45$

$= 141$

Average velocity $= \dfrac{141 - 45}{8 - 0}$

$= 12$

The average velocity from $t = 0$ to $t = 8$ is 12 m/s.

(2) f. Find the acceleration at any time.

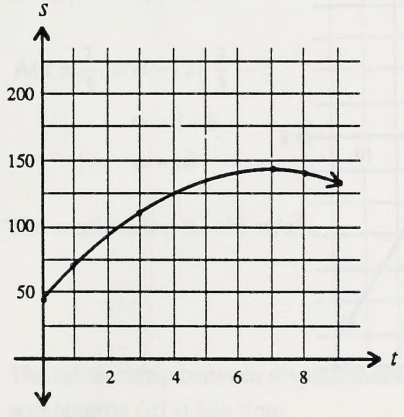
$a = D_t v = -4 \text{ m/s}^2$

(2) g. Find the acceleration at $t = 4$.

$a = -4 \text{ m/s}^2$

(2) h. Sketch the graph of s as a function of t.

t	0	1	3	7	8	9
s	45	71	111	143	141	135

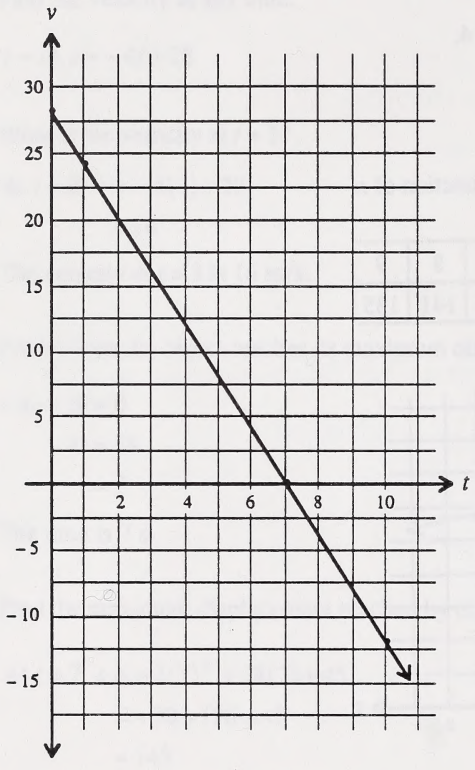


(2) i. Describe the direction of the motion at $t = 3$ s, 7 s, and 9 s.

At $t = 3$, the direction of the motion is positive.
At $t = 7$, the object stops. It has reached its maximum displacement.
At $t = 9$, the direction of the motion is negative. The object is moving in the opposite direction.

(2) j. Sketch the graph of v as a function of t.

t	0	1	7	10
v	28	24	0	– 12



(2) k. Describe both v and a.

The velocity of the object is decreasing from $t = 0$ to $t = 7$. At $t = 7$, the object stops and then moves in the opposite direction. The acceleration is a negative constant. It is actually a constant deceleration until $t = 7$ s. After $t = 7$ s, it accelerates but in the opposite direction; therefore, the acceleration is always $-4\ \text{m/s}^2$.

(5) 2. An object is moving in a straight line. The distance in metres from a fixed point is given by $s = -2t^3 - 4t^2 + 6t + 4,\ t \geq 0$. Determine the acceleration when the velocity is -2 m/s.

$v = D_t s = -6t^2 - 8t + 6$

$-6t^2 - 8t + 6 = -2$

$-6t^2 - 8t + 8 = 0$

$3t^2 + 4t - 4 = 0$

$(3t - 2)(t + 2) = 0$

$3t - 2 = 0, \quad t + 2 = 0$

$t = \frac{2}{3} \qquad t = -2 \quad (t \text{ cannot be negative.})$

$a = D_t v = -12t - 8$

$\text{At } t = \frac{2}{3},\ a = -12\left(\frac{2}{3}\right) - 8$

$= -8 - 8$

$= -16$

The acceleration is -16 m/s^2.

(4) 3. The relationship between displacement (s) and velocity (v) is given by $\frac{1}{3}s^2 - 4v^2 = 12$. Find the acceleration (a) at any time.

$D_t\left(\frac{s^2}{3}\right) - D_t\left(4v^2\right) = D_t(12)$

$\frac{2}{3}s\frac{ds}{dt} - 8v\frac{dv}{dt} = 0$

$\frac{2s}{3}(v) - 8v(a) = 0$

$a = \frac{2vs}{3} \div (8v)$

$= \frac{s}{12}$

The acceleration is $\frac{s}{12}$.

④ 4. The relation between distance (s) and velocity (v) is given by $v = \frac{150s}{3+s}$. Find the acceleration in terms of s.

$$a = D_t v = \frac{(3+s)D_t(150s) - (150s)D_t(3+s)}{(3+s)^2}$$

$$= \frac{(3+s)(150)v - (150s)v}{(3+s)^2}$$

$$= \frac{450v + 150vs - 150vs}{(3+s)^2}$$

$$= \frac{450v}{(3+s)^2}$$

$$= \frac{450\left(\frac{150s}{3+s}\right)}{(3+s)^2}$$

$$= \frac{67\,500s}{(3+s)^3}$$

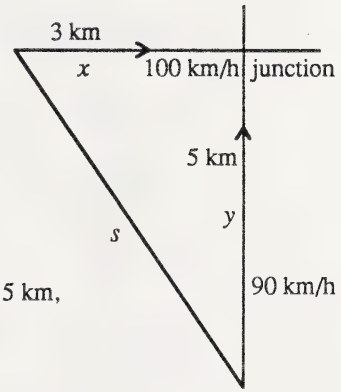
The acceleration in terms of s is $\frac{67\,500s}{(3+s)^3}$.

Topic 2

_____ marks

Topic 3: Rates of Change

(6) 1. Two straight highways are perpendicular to each other. One car is 3 km from the intersection and moving at 100 km/h toward the junction. On the other highway another car is 5 km from the intersection and moving at 90 km/h toward the junction. At what rate is the distance between the two cars changing at that instant?



$$s^2 = x^2 + y^2$$

$$D_t\left(s^2\right) = D_t\left(x^2\right) + D_t\left(y^2\right)$$

$$2s\,D_t s = 2x\,D_t x + 2y\,D_t y$$

$$D_t s = \frac{x\,D_t x + y\,D_t y}{s}$$

Since $D_t x = 100$ km/h, $D_t y = 90$ km/h, $x = 3$ km, and $y = 5$ km,

$$s = \sqrt{3^2 + 5^2}$$

$$= \sqrt{34}.$$

$$\therefore D_t s = \frac{3(100) + 5(90)}{\sqrt{34}}$$

$$= \frac{750}{\sqrt{34}}$$

$$\doteq 128.62$$

The distance between the two cars is changing at a rate of about 128.62 km/h.

(6) 2. The height of an equilateral triangle is increasing at the rate of 3 cm/min. How fast is its area changing when h is 5 cm?

$$h^2 = x^2 - \left(\frac{x}{2}\right)^2$$

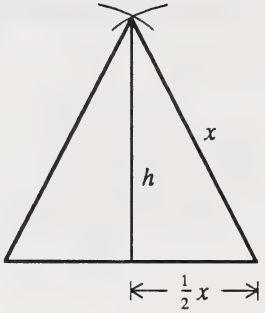
$$= x^2 - \frac{x^2}{4}$$

$$= \frac{4x^2 - x^2}{4}$$

$$= \frac{3x^2}{4}$$

$$h = \frac{\sqrt{3}x}{2}$$

$$\therefore x = \frac{2h}{\sqrt{3}}$$



$$\text{Area } (A) = \frac{1}{2}xh$$

$$A = \frac{1}{2}\left(\frac{2h}{\sqrt{3}}\right)h$$

$$= \frac{h^2}{\sqrt{3}}$$

$$D_t A = \frac{2h}{\sqrt{3}} \times \frac{dh}{dt}$$

$$\text{Since } \frac{dh}{dt} = 3 \text{ cm/min}, \quad D_t A = \frac{2}{\sqrt{3}}(5)(3)$$

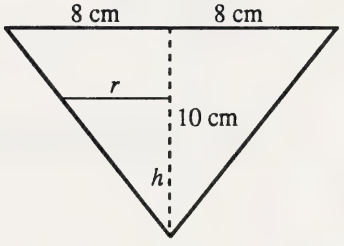
$$= \frac{30}{\sqrt{3}}$$

$$= 10\sqrt{3}$$

$$\doteq 17.321$$

The area of the triangle is changing at about 17.321 cm^2/min.

⑦ 3. A coffee filter has the shape of an inverted cone. The radius is 8 cm at the top and the height is 10 cm. Water is drained from the filter at the rate of 25 cm^3/s. How fast is the depth of the water decreasing when it is 5 cm deep?



$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{r}{8} = \frac{h}{10}$$

$$r = \frac{8h}{10}$$

$$= \frac{4h}{5}$$

$$V = \frac{1}{3}\pi\left(\frac{16h^2}{25}\right)h$$

$$= \frac{16\pi}{75}h^3$$

$$D_t V = \frac{16\pi}{75}\left(3h^2\right)D_t h$$

At $h = 5$ cm, $\frac{dV}{dt} = 25\ cm^3/s$.

$$\frac{dh}{dt} = \frac{(25)(75)}{16\pi(3)(25)}$$

$$= \frac{25}{16\pi}$$

$$\doteq 0.497$$

The depth of the water is decreasing at about 0.497 cm/s.

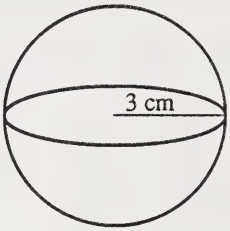
(5) 4. The radius of a spherical balloon is increasing at the rate of 2 cm/s when the radius is 3 cm. At what rate is the air pumped in?

$$V = \frac{4}{3}\pi r^3$$

$$D_t V = \frac{4}{3}\pi\left(3r^2\right)D_t r$$

$$= 4\pi(3)^2(2)$$

$$= 72\pi$$



The air is pumped at a rate of 72π cm^3/s.

(6) 5. The curve $y = -3x^2 + 2x - 7$ is the path of a moving point. If the x-coordinate of this point is increasing at the rate of two units per second when $x = 5$, at what rate is the y-coordinate changing?

$$\frac{dy}{dt} = -6x\frac{dx}{dt} + 2\frac{dx}{dt}$$

At $x = 5$, $\frac{dx}{dt} = 2$

$$\frac{dy}{dt} = -6(5)(2) + 2(2)$$

$$= -60 + 4$$

$$= -56$$

The y-coordinate is changing at -56 units/s.

Topic 3

_____ marks